A DISTURBING SUPERTASK

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Abstract. This paper examines the consistency of ω-order by means of a supertask that functions as a supertrap for the assumed existence of ω-ordered collections, which are simultaneously complete (as is required by the Actual infinity) and uncompletable (because no last element completes them).

As Cantor himself proved [2], [3], ω-order is a formal consequence of assuming the existence of denumerable sets as complete totalities. Although it is hardly recognized, to be ω-ordered means to be both *complete* and *uncompletable*. In fact, the Axiom of Infinity states the existence of complete denumerable totalities, the most simple of which are ω-ordered, i.e. with a first element and such that each element has an immediate successor. Consequently, there is not a last element that completes ω-ordered totalities. To be complete and uncompletable may seem a modest eccentricity in the highly eccentric infinite paradise of our days, but its simplicity is just an advantage if we are interested in examining the formal consistency of ω-order. In addition, ω is the first transfinite ordinal, the one on which all successive transfinite ordinals are built up. This magnifies the interest of examining its formal consistency. The short discussion that follows is based on a supertask conceived to put into question just the ability of being complete and uncompletable that characterizes ω-order.

1. The last disk: a disturbing supertask

Consider a hollow cylinder C and an ω-ordered collection of identical disks $\langle d_i \rangle_{i \in \mathbb{N}}$ such that each disk d_i fits exactly within the cylinder. For the sake of clarity, we will assume that a disk d_0 is initially placed inside the cylinder, although this is irrelevant to our discussion.

Let a_i be the action of replacing disk d_{i-1} inside the cylinder by its immediate successor disk d_i, which is accomplished by placing d_i completely within the cylinder. Consider the ω-ordered sequence of actions $\langle a_i \rangle_{i \in \mathbb{N}}$ and assume that each action a_i is carried out at instant t_i, being t_i an element of the ω-ordered sequence of instants $\langle t_i \rangle_{i \in \mathbb{N}}$ in the real interval $[t_a, t_b)$ such that:

$$\lim_{i \to \infty} t_i = t_b \tag{1}$$

Let S_ω be the supertask of performing the ω-ordered sequence of actions $\langle a_i \rangle_{i \in \mathbb{N}}$. We will impose a restriction (restriction R) to S_ω: for all

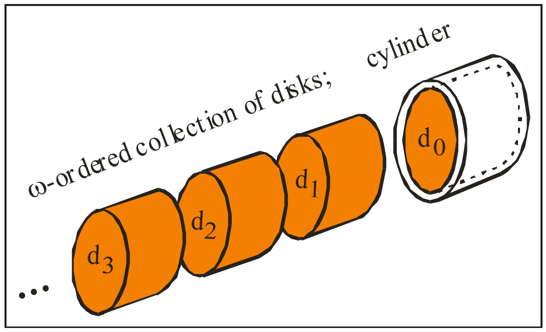

Figure 1: Cylinder C and the ω-ordered collection of disk $\langle d_i \rangle_{i \in \mathbb{N}}$.

i in $\mathbb{N}$, a_i will be performed if, and only if, it leaves the cylinder completely occupied by disk d_i. It is immediate to prove that all actions $\langle a_i \rangle_{i \in \mathbb{N}}$ observe this restriction: in fact it is clear that a_1 observes R because it leaves the cylinder completely occupied by disk d_1. Assume the first n actions observe R. It is quite clair that a_{n+1} also observes R: it leaves the cylinder completely occupied by disk d_{n+1} because, by definition, it consists just in placing d_{n+1} completely within the cylinder. Consequently all actions $\langle a_i \rangle_{i \in \mathbb{N}}$ observe restriction R. Consider now the one to one correspondence f defined as:

$$f(t_i) = d_i \tag{2}$$

It immediately proves that, being t_b the limit of the ω-ordered sequence $\langle t_i \rangle_{i \in \mathbb{N}}$, at t_b all actions a_i have been performed. Thus at t_b supertask S_ω has been completed.

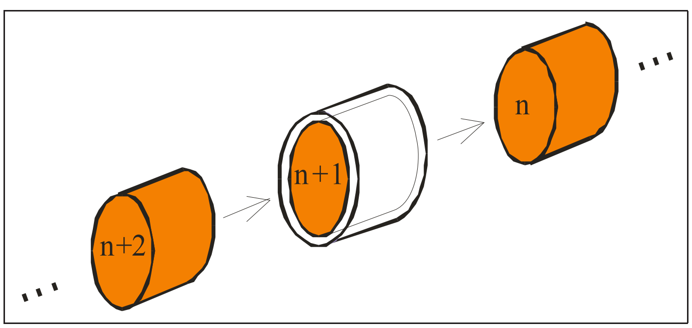

Figure 2: The $(n+2)$-th action of supertask S_ω about to be performed.

As is well known, we cannot infer the final state of the devices involved in a supertask from the successive performed actions [1]. But, if supertasks do not destroy the world nor change its nature, we will have the opportunity of deriving some interesting conclusions on the consistency of supertasks and ω-order just from the very nature of the involved devices.

2. Discussion

With respect to the possibilities of being occupied by disks $\langle d_i \rangle_{i\in\mathbb{N}}$, the cylinder C can only exhibit one, and only one, of the following three alternatives:

(1) Occupied by two disks partially placed within it.
(2) Occupied by one disk completely or partially placed within it.
(3) Occupied by no disk.

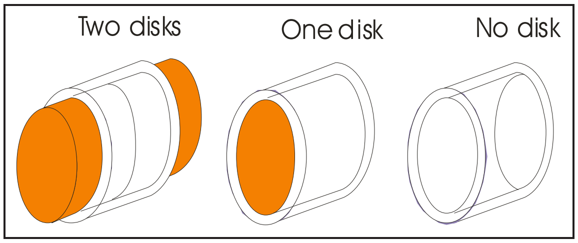

Figure 3: Possibilities for the cylinder C of being occupied by disks $\langle d_i \rangle_{i\in\mathbb{N}}$.

It is worth noting this conclusion does not depend on the number of performed actions but on the fact of being a hollow cylinder in whose interior each disk d_i exactly fits. Consequently it must hold at any instant, before S_ω, along S_ω and after S_ω, which includes the precise instant t_b at which supertask S_ω has been completed.

It is immediate to prove that at t_b, once completed S_ω, no disk can be within the cylinder. In fact, if one disk[1] were within the cylinder at t_b only a finite number of disks would have been placed inside it from t_a to t_b: just the disk within it at t_b and all its predecessors, the number of which is always finite for every element of any ω-ordered sequence. Consequently, at t_b there would remain an infinite number of disks to be placed inside the cylinder: all the infinitely many successors of the disk placed inside it at t_b. Accordingly, if C contains one disk at t_b then only a finite number of actions a_i would have been carried out. Note this conclusion does not depend on the number of performed actions but on having a finite number of predecessors and an infinite number of successors, an attribute of every element of an ω-ordered sequence. It is therefore impossible that cylinder C contains any disk at t_b if supertask S_ω has been completed. Thus, the completion of $\langle a_i \rangle_{i\in\mathbb{N}}$ leaves the cylinder empty of disks (in spite of the fact that none of the countably many performed actions a_i left it empty). So, the completion of $\langle a_i \rangle_{i\in\mathbb{N}}$ violates the restriction R that each one of the performed actions a_i strictly observed!

[1]The same argument would be applied if the cylinder contains two disks.

Infinitists claim that although, in fact, no particular action a_i leaves the cylinder empty, the completion of all of them does it. We must, therefore, analyze this possibility. There are two alternatives regarding the completion of the ω-ordered sequence of actions $\langle a_i \rangle_{i \in \mathbb{N}}$:

(1) The completion is an additional (ω +1)-th action.
(2) The completion is not an additional (ω +1)-th action. It simply consists in performing each one of the countably many actions $\langle a_i \rangle_{i \in \mathbb{N}}$, and only them.

Let us examine the first alternative[2]. The assumed (ω +1)-th action can only occurs at t_b (or even later) because at any instant prior to t_b there still remain infinitely many actions to be performed (in an ω-ordered sequence of actions can never remain a finite number n of actions to be performed because they would be the impossible lasts n terms of an ω-ordered sequence). Whatever be the instant we consider, if it is prior to t_b, there will remain infinitely many actions to be performed and only a finite number of them will have been performed[3]. Thus, at t_b (or later) the cylinder has to be occupied by a disk; otherwise, if the cylinder were empty at t_b, the assumed (ω +1)-th action, which occur at t_b (or later) and consists just in leaving the cylinder empty, would not be the cause of leaving the cylinder empty as it is assumed to be. We have, therefore, a disk inside the cylinder at t_b. And, for the same reasons above, this is impossible if S_ω has been completed: the disk inside the cylinder would be proving that only a finite number of actions would have been carried out. Thus, the first alternative is impossible.

We will examine, then, the second one. According to this alternative the cylinder becomes empty as a consequence of having completed the countably many actions a_1, a_2, a_3, ..., and only them. Thus, either the successive actions have an accumulative effect capable of leaving finally the cylinder empty, or the completion has a sort of sudden final effect on the cylinder as a consequence of which it results empty. We can rule out this last possibility for exactly the same reasons we have ruled out the (ω +1)-th additional action (it must take place at t_b and then at t_b there would be a disk inside the cylinder proving that at t_b only a finite number of actions would have been performed). The only possibility is, therefore, that cylinder C becomes empty as a consequence of a certain accumulative effect of the successively performed actions. Let e_i be the volume inside the cylinder which is not occupied by disk d_i once d_i is placed within the cylinder by a_i, i.e. the empty volume inside C once d_i has been placed in C. According to the above definition of a_i we

[2] Some infinitists support it.

[3] As far as I know, this unaesthetic and huge asymmetry is never mentioned in supertask literature.

will have:

$$e_i = 0, \ \forall i \in \mathbb{N} \tag{3}$$

Let us then define series $\langle s_i \rangle_{i \in \mathbb{N}}$ as:

$$s_i = e_1 + e_2 + \cdots + e_i, \ \forall i \in \mathbb{N} \tag{4}$$

The i-th term s_i of this series represents, therefore, the empty volume inside the cylinder once performed the firsts i actions. Evidently we will have:

$$s_i = 0, \ \forall i \in \mathbb{N} \tag{5}$$

$\langle s_i \rangle_{i \in \mathbb{N}}$ is therefore a series of constant terms. Thus we have:

$$\lim_{i \to \infty} s_i = 0 \tag{6}$$

And then:

$$\lim_{i \to \infty} s_i = \sum_{i=1}^{\infty} s_i = 0 \tag{7}$$

Therefore, once completed the ω-ordered sequence of actions $\langle a_i \rangle_{i \in \mathbb{N}}$, and having each a_i left an empty volume $e_i = 0$ inside the cylinder, the resulting empty volume inside the cylinder is also null. Cylinder C cannot, therefore, results empty as a consequence of an accumulative effect of the successively performed actions a_i. Therefore, the completion of the ω-ordered sequence of actions $\langle a_i \rangle_{i \in \mathbb{N}}$ does not leave the cylinder empty. In consequence, supertask S_ω leads to a contradiction: the completion of $\langle a_i \rangle_{i \in \mathbb{N}}$ leaves and does not leave the cylinder empty of disks.

We will consider now the finite versions of S_ω. For this let n be any natural number and $\langle d_i \rangle_{1 \leq i \leq n}$ the finite collection of the first n disks of $\langle d_i \rangle_{i \in \mathbb{N}}$. As in the case of supertask S_ω, let a_i be the action of replacing disk d_{i-1} of $\langle d_i \rangle_{1 \leq i \leq n}$ within C with its immediate successor disk d_i at instant t_i. Let S_n be the task of performing the finite sequence of actions $\langle a_i \rangle_{1 \leq i \leq n}$. It is immediate to prove that at t_n all these actions will have been performed and the cylinder will finally contains the last disk d_n placed within it. No contradiction arises here. And this holds for every natural number: S_n is consistent for every n in $\mathbb{N}$. Only S_ω is inconsistent. But the only difference between S_ω and $S_{n, \forall n \in \mathbb{N}}$ is just the ω-order of S_ω. The contradiction with S_ω can only derive from this type of infinite ordering, and then from the Axiom of Infinity, of which it is a formal consequence. Thus, the argument above is not on the impossibility of a particular supertask, but on the inconsistency of ω-order. Being complete and uncompletable could be, after all, a formal inconsistency rather than an eccentricity of the first transfinite ordinal.

References

[1] Paul Benacerraf, *Tasks, Super-tasks, and Modern Eleatics*, Journal of Philosophy **LIX** (1962), 765–784. 1

[2] Georg Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, Mathematishe Annalen **XLIX** (1897), 207 – 246. (document)

[3] ———, *Contributions to the founding of the theory of transfinite numbers*, Dover, New York, 1955. (document)